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# GRAVITATIONAL RADIATION FROM BINARY SYSTEMS

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UNPUBLISHED PRELIMINARY DATA

"Gravitational Radiation from Binary Systems"
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## Gravitational Radiation from Binary Systems:

Two bodies of masses $m_1$ and $m_2$ attracting each other according to Newton's law of gravitation move in circular orbits about their common centre of mass. The system acts as a source of gravitational waves which reduce the total energy of the system at a rate $\left(-\frac{dE}{dt}\right)$. As a result, the separation r between the two masses progressively decreases while the angular velocity of rotation $\omega$ increases*. We investigate the variation of these quantities and apply the results to astronomical objects.

### Rate of Energy Release:

The rate at which gravitational energy is released by any time-dependent mass distribution can be calculated on the basis of linearised field equations (See Landau & Lifshitz "Classical Theory of Fields", Ch. 11). The expression thus obtained is

$$\frac{-dE}{dt} = \frac{G}{45c^5} (\dddot{D}ij)^2 \qquad i, j = 1, 2, 3 \tag{1}$$

where G = Gravitational Constant = $6.67 \times 10^{-8}$ $cm^3$ $gm^{-1}$ $sec^{-2}$

c = Velocity of Light = $3 \times 10^{10}$ cms/sec

and $$Dij = \int \rho \, (3x^i x^j - \delta_{ij} \; x_k^2) \, dv \tag{2}$$

where $\rho$ = Density of mass distribution.

We denote $\left(\frac{-dE}{dt}\right)$ in future by Luminosity L in view of the fact that we would eventually be dealing with astronomical bodies.

### Evaluation of $(\dddot{D}ij)^2$:

One can always compute the various components Dij individually as functions of time and evaluate $(\dddot{D}ij)^2$. However, when rotation of rigid bodies is

* The frequency of the emitted gravity waves is $\omega_g = 2\,\omega$ as we are dealing with quadrupole radiation.

involved, a simple and elegant expression can be obtained as follows:

We can express the 3 x 3 matrix $D = [Dij]$ at time t as

$D(t) = A(t)D^{o}A^{-1}(t)$

where $D^{o}$ = the initial value of matrix D at t = o

$A(t)$ = rigid rotation operator which can be represented as a 3 x 3 real orthogonal matrix (See Corben and Stehle "Classical Mechanics", Ch. 9 or Goldstein "Classical Mechanics", Ch. 4).

We note $A^{-1}(t) = A^{T}(t)$

and $A(t) = e^{\Omega t}$ where $\Omega$ is the infinitesimal rotation matrix. (A specific form for $\Omega$ will be assumed below.)

Also $\Omega^{T} = -\Omega$

It follows then, $\frac{dA}{dt} = \Omega A$; $\frac{dA^{T}}{dt} = A^{T}\Omega^{T} = -A^{T}\Omega$.

Differentiating the equation for D(t) with respect to time,

$$\dot{D}(t) = \Omega AD^{o}A^{T} - AD^{o}A^{T}\Omega$$

$$\ddot{D}(t) = \Omega^{2}AD^{o}A^{T} - 2\,\Omega AD^{o}A^{T}\Omega + AD^{o}A^{T}\Omega^{2}$$

$$\dddot{D}(t) = \Omega^{3}AD^{o}A^{T} - AD^{o}A^{T}\Omega^{3} + 3(\Omega AD^{o}A^{T}\Omega^{2} - \Omega^{2}AD^{o}A^{T}\Omega)$$

In case of uniform rotation the final result for $\dddot{D}ij^{2}$ would be independent of time, so that we can replace A(t) by its initial value A(o) = 1

$$\text{Then } \dddot{D} = (\Omega^{3}D^{o} - D^{o}\Omega^{3}) + 3(\Omega D^{o}\Omega^{2} - \Omega^{2}D^{o}\Omega)$$

For uniform rotation about Z axis with angular velocity $\omega$ one can write $\Omega$ specifically as:

$$\Omega = \omega\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix};\quad \Omega^{2} = \omega^{2}\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix};\quad \Omega^{3} = \omega^{3}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = -\Omega\,\omega^{2}$$

Making use of the last relation and denoting $\frac{\Omega}{\omega} = \Lambda$, we get

$$\dddot{D} = \omega^{3}(D^{o}\Lambda - \Lambda D^{o})(1 - 3\Lambda^{2})$$

Further, if at t = 0 the principal axes of the rigid body coincide with the spatial axes of reference, $D^{o}$ would be diagonal,

i. e., $D^o = \begin{pmatrix} D^o_{11} & 0 & 0 \\ 0 & D^o_{22} & 0 \\ 0 & 0 & D^o_{33} \end{pmatrix}$

With this stipulation and utilizing the specific form of $\Lambda$ for Z axis quoted already, the expression simplifies to

$$\dddot{D} = \omega^3 \begin{pmatrix} 0 & 4(D^o_{11} - D^o_{22}) & 0 \\ 4(D^o_{11} - D^o_{22}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Finally,

$$\dddot{D}ij^2 = 2\dddot{D}^2_{12} = 32(D^o_{11} - D^o_{22})^2 \omega^6 \tag{3}$$

Using the relationship

$$D_{ij} = \delta_{ij} I_{kk} - 3I_{ij} \tag{4}$$

between Dij and the moment of inertial tensor

$$I_{ij} = \int (r^2 \delta_{ij} - X_i X_j) \rho \, dV \tag{5}$$

We can also rewrite the formula (1) for power loss to gravitational radiation by a body rotating rigidly about a principle axis as

$$L = -\frac{dE}{dt} = \frac{32}{5} \frac{G}{c^5} (I_2 - I_1)^2 \omega^6 \tag{6}$$

Applications of the Formula:

a. Binary Systems

For two point masses revolving around their common center of mass, it is evident that the moment of inertia $I_1$, about an axis through the masses is zero. The well known formula $J = \mu r^2 \dot{\theta}$ for their angular momentum then gives $I_2 = \mu r^2$. Here $\mu = m_1 m_2 (m_1 + m_2)^{-1}$ is the reduced mass and $r = |x_1 - x_2|$ is their separation. The total power emitted in gravitational radiation, the gravitational luminosity, is then

$$L_B = \frac{32}{5} \frac{G}{c^5} (\mu r^2 \omega^3)^2 \tag{7}$$

b. Ellipsoid

For an ellipsoid of mass m with semi-axes a, b, c the moments of inertia are

$$I_1 = \frac{m}{5} \quad (b^2 + c^2)$$

$$I_2 = \frac{m}{c} \quad (c^2 + a^2)$$

This leads to a luminosity in gravitational radiation of

$$L_e = \frac{32}{125} \frac{G}{c^5} m^2 (a^2 - b^2)^2 \omega^6 \tag{8}$$

<u>Natural Unit for Gravitational Radiation</u>:

The kinetic energy of the binary system is given by

$\frac{1}{2} \mu r^2 \omega^2$

If the period of revolution is $T = 2\Pi / \omega$ we can think of the "kinetic" luminosity (or "circulating power") of the system,

$$L_k = \frac{\text{Kinetic Energy}}{\text{period}} = \frac{1}{2} \mu r^2 \omega^2 \cdot \frac{\omega}{2\Pi} = \frac{1}{4\Pi} (\mu r^2 \omega^3)$$

$4\Pi L_k = \mu r^2 \omega^3$, which is precisely the quantity, the square of which appears in the expression for $L_B$. We have then

$$L_B = \frac{16\Pi^2 \times 32G}{5c^5} \quad (L_k)^2$$

Clearly, the constant multiplying $L_k^2$ must be of the dimension $L^{-1}$, so that we define the Characteristic Gravitational Luminosity

$$Lg = \frac{5c^5}{16 \times 32\pi^2 G} = 3.6 \times 10^{56} \text{ ergs/sec} = 2 \times 10^2 \text{ M}_\odot c^2\text{/sec} \tag{9}$$

where $M_\odot$ = solar mass = $2 \times 10^{33}$ gms.

It is also evident that one can define a natural unit for gravitational radiation in terms of the Universal Constants G and c:

$$Ln = \frac{c^5}{G} = 3.64 \times 10^{59} \text{ ergs/sec} = 2.02 \times 10^5 \text{ M}_\odot c^2\text{/sec} \tag{10}$$

In the formula

$$\frac{L_B}{Lk} \cdot \frac{Lg}{Lk} = 1 \tag{11}$$

one has a readily available means of estimating the energy radiated by the binary system in terms of Lk. Further, whenever Lk is comparable in magnitude to Lg, so will $L_B$ be. (This way of stating the radiation formula was first given by Dyson.)

Variation of $\omega$ and r:

The balance between gravitational attraction and centrifugal force, viz., the equation

$\dfrac{Gm_1m_2}{r^2} = \dfrac{m_1m_2}{m_1+m_2}\ \omega^3 r$ yields $\omega$ as a function of r:

$$\omega^2 = G(m_1+m_2)/r^3$$

Substituting this in $L_B$ we obtain,

$$L_B = \frac{32G^4(m_1m_2)^2(m_1+m_2)}{5c^5} \cdot \frac{1}{r^5} = \frac{A}{r^5} \tag{12}$$

Next, the total energy of the system is:

$E = V + T$; Kinetic Energy $T = -\frac{1}{2}V$ by virial theorem so that

$$E = \tfrac{1}{2}V = -\tfrac{1}{2}\,\frac{Gm_1m_2}{r}$$

Differentiating both sides of the equation with respect to time,

$$\frac{dE}{dt} = \frac{1}{2}\ \frac{Gm_1m_2}{r^2}\ \frac{dr}{dt}$$

Equating $\dfrac{dE}{dt}$ to $-L_B$ above,

$$\frac{dr}{dt} = -\frac{64G^3m_1m_2(m_1+m_2)}{5c^5} \cdot \frac{1}{r^3} = -B/r^3 \tag{13}$$

This equation can be readily integrated to give,

$$r^4 = a^4 - 4Bt \tag{14}$$

where a = initial separation, i. e., at the instant from which time t is counted.

We can think of the "life time" of the system which is the time taken for the separation to fall down from any given initial value "a" to zero. Clearly, life time

$$T = a^4/4B \tag{15}$$

Before taking up the application of these formulae we may note in passing the following fact.

Suppose we take $m_1 = m_2 = m$. Then,

$$L_B = \frac{64G^4m^5}{5c^5} \times \frac{1}{r^5} = \frac{2}{5}\frac{c^5}{G}\left(\frac{2Gm}{c^2r}\right)^5 \tag{16}$$

When the separation = Schwarzschild Radium, i. e., $r = r_s = 2Gm/c^2$

$$(L_B)_{r=r_s} = 2/5\ c^5/G = 2/5\ Ln = 1.46 \times 10^{59} \text{ ergs/sec}$$

which is a constant independent of the mass m.

Application to Astronomical Bodies:

In the discussion to follow we shall take $m_1 = m_2 = 1M_\odot$

The two constants A and B will then be

$$A = 1.66 \times 10^{87}; \quad Bo = 2.5 \times 10^{27} \text{ in c. g. s. units.}$$

And $T = a^4 \times 10^{-28}$ secs.

Consider the life time for three characteristic values of initial separation:

1) The median value for the separation of binary stars is $\sim$ 20AU$\cong$ $3 \times 10^{14}$ cm (1AU = $1.496 \times 10^{13}$ cm) which is of the same order of magnitude as the distance of planets from the sun.

For this separation, we find

$$T = 8.1 \times 10^{29} \text{ secs} \cong 2.5 \times 10^{22} \text{ years } (1 \text{ year} = 3.155 \times 10^7 \text{ secs})$$

which is an extremely long period of time (compare with the age of earth $4.5 \times 10^9$ years). Luminosity at this separation is only about $10^{17}$ ergs/sec. In short, the effect of gravitational radiation on binary stars is negligible.

2) For $a \sim$ radius of an average star ($10^{11}$ cms)

$T \simeq 3 \times 10^8$ years; $L_B = 10^{32}$ ergs/sec.

3) Binary resulting from the symmetric fission of an average star:

Suppose a spherical star of radius $R_i$ and mass $M_i$ spinning at a frequency $\omega_i$ breaks into two equal parts of mass $M_f$ each ($M_i = 2M_f$) separated by a distance 'a' and revolving at a frequency $\omega_f$. Assuming the angular momentum is conserved during this process, we can write

Angular momentum, $J = I_i \omega_i = I_f \omega_f$

where $I_i$ = Moment of Inertia of the original star = $\frac{2}{5}$ MiRi$^2$

$I_f$ = Moment of Inertia of the binary = $\frac{1}{2} M_f a^2$ .

Further, for the binary $\omega_f^2 = \frac{2GM_f}{a^3}$ on account of the equality of gravitational attraction and centrifugal force.

Therefore conservation of J yields

$$\tfrac{1}{2} M_f a^2 \times \left(\frac{2M_f G}{a^3}\right)^{\frac{1}{2}} = \frac{2}{5} (2M_f) \, Ri^2 \omega i$$

$$\text{or } (a)^{\frac{1}{2}} = \frac{8}{5} \left(\frac{1}{2M_f G}\right)^{\frac{1}{2}} \quad Ri^2 \omega i$$

For an "average" star $R \sim 10^{11}$ cms; $T = \frac{2\Pi}{\omega} = 1$ day

The fission of such a star leads therefore to

$a \simeq 2.5 \times 10^9$ cms.

For $a \sim 10^9$ cms, $T = 10^8$ secs = 3 years.

and $L_B \sim 10^{42}$ ergs/sec as compared with the solar luminosity $L_\odot$ = $3.9 \times 10^{33}$ ergs/sec.

We may conclude that a binary system that has evolved out of the fission of a star as above can act as a powerful source of gravitational radiation, the rate of radiation increasing to values much higher than the initial luminosity within a short period of time. We record below three stages during the life of the binary system under consideration:

| | | | |
|---|---|---|---|
| r cms | $10^9$ | $3 \times 10^8$ | $10^6$ |
| L ergs/sec | $10^{42} \gg L_\odot$ | $2 \times 10^{43} \sim L^*$galaxy | $3.6 \times 10^{56} = L_g$ |
| $\nu = \frac{\omega}{2\Pi}$ cps | 0.1 | 0.3 | $10^3$ |
| T years | 3 | 0.3 | $10^{-12}$ |

However, as the separation is decreased in our calculations, a limiting value of r is reached beyond - and possibly around - which the foregoing considerations are no longer valid. In the first place, if the velocity of each mass is computed we find

$$v = \tfrac{1}{2}\omega r = \tfrac{1}{2}\left(\frac{2MG}{r^3}\right)^{\frac{1}{2}} \times r = \left(\frac{1}{4} \; 2MG/r\right)^{\frac{1}{2}}$$

$$\frac{v}{c} = \left(\frac{1}{4} \; (2MG/c^2)/r\right)^{\frac{1}{2}} = \left(\frac{rs/4}{r}\right)^{\frac{1}{2}}$$

As $r \rightarrow r_s/4$ , $v \rightarrow c$

Secondly, at such small separations the gravitational field would be so strong that one may expect the linear theory to be wholly inadequate. We estimate a limit as follows:

As the two masses become tightly bound, they form a single system at mass

* The value corresponds to the luminosity of the spiral galaxy in Andromeda.

2M and Schwarzschild radius $2r_s$. As the Schwarzschild spheres of the separate masses, each of radius $r_s$, begin to touch the region which will be the common Schwarzschild sphere of diameter $4r_s$, non-linear effects must already be very important. Thus for $r < 6r_s$ the formulas here are almost certainly incorrect even in order of magnitude. The total energy radiated up to this point will not exceed the linearized theory estimate which is given by the Newtonian binding energy

$$\frac{\frac{1}{2}GM^2}{(6r_s)} = \frac{1}{48}\ (2Mc^2)$$

Whether more than this 2% fraction of the rest mass can be radiated away is a question the linearized theory cannot answer.

Comparison with Radiation from Spheroidal Stars and Effect of Density on Radiation Rate:

Consider a spheroidal star (semi axes a, b, b) spinning about Z axis with angular velocity $\omega$. Let us assume that density $\rho$ = density of our average spherical star and mass M = Mass of the spherical star = $2M_\odot$.

$$\text{Then } 2M_\odot = \frac{4}{3}\,\Pi R^3\rho = \frac{4}{3}\,\Pi a b^2 \rho$$

so that $R^3 = ab^2$.

If we take $a = 2R$, $b = R/\sqrt{2}$, with $R = 10^{11}$ cms, we can calculate L from the expression already derived, viz.,

$$Le = \frac{32G}{125c^5}\ (a^2 - b^2)^2\ M^2\omega^6$$

We find $Le = 5.2 \times 10^{26}$ ergs/sec which is indeed a small value and corresponds to $L_B$ with $r = 10^{12}$ cms.

We next examine for what values of a and b the spheroid can produce.

$Le = 1.66 \times 10^{42}$ ergs/sec $= L_B$ for $r = 10^9$ cms.

Retaining the above assumptions about M and $\rho$ it can be shown that the a value of $10^{42}$ ergs/sec can be obtained for Le if a = 2 x $10^{30}$ cms and b = 22 cms!

Nevertheless, we must note that during fission of the spherical star, the density of matter increased by an enormous amount. For instance, if each component of the binary has a radius a/4 (a = $10^9$ cms), the density would be $\sim 3 \times 10^7$ gms/cc in contrast to the initial value of 1 gm/cc. If the spheroidal star is allowed to have such a high density and if the assumption made for the fission are retained, viz., angular momentum and mass of the spheroidal star are the same as those of the spherical star, the energy release would once again be of high order of magnitude.

The equations governing the calculations would now be

$$J = \frac{2}{5} MR_i^2 \, \omega_i = \frac{M}{5} (a^2 + b^2) \, \omega_e \; ; \quad M = \frac{4\Pi}{3} ab^2 \rho$$

where $M = 2M_\odot$ and $\rho = 3 \times 10^7$ gms/cc.

Assuming the same ratio $\frac{a}{b} = 2\sqrt{2}$ as before, we find $a \sim 6 \times 10^8$ cms.

And the luminosity

$Le \cong 2.8 \times 10^{44}$ ergs/sec

This illustrates the important role played by density in the emission of gravitational energy.

Computations:

We list here the formulae used in preparing the figure at the end. Constants have been evaluated for $m_1 = m_2 = M_\odot = 2 \times 10^{33}$ gms

a) $L_B = \frac{-dE}{dt} = Ao/r^5 = \frac{1.66 \times 10^{87}}{r^5}$ ergs sec$^{-1}$

b) Frequency of rotation $\nu = \frac{\omega}{2\Pi} = 2.601 \times 10^{12}/r^{3/2}$

Note: The emitted frequency, as already pointed out, is twice this value.

c) $T = r^4/4Bo = r^4 \times 10^{-28}$ secs $= r^4 \times 3.171 \times 10^{-36}$ years.

d) The maximum distance 'd' of the source from earth for which detection is possible:

If the minimum detectable flux is F ergs $sec^{-1}$ $cm^{-2}$ then $F = L_B/4\pi d^2$

For acoustic vibrations of earth's modes (See Forward, Weber & Zipoy, Nature Vol 189, p 473; 1961):

Frequency of lowest mode $\sim$ 1 cycle per hour; F $\sim$ 5 ergs $sec^{-1}$ $cm^{-2}$

For the detector being built at the University of Maryland which will directly respond to gravitational waves (See Weber "General Relativity and Gravitational Waves"):

Frequency $\sim$ 1600 cps; F $\sim$ 10 ergs $sec^{-1}$ $cm^{-2}$

In plotting the figure, F has been taken as 1 erg $sec^{-1}$ $cm^{-2}$ for convenience, so that

$d = (L_B/4\pi)^{\frac{1}{2}}$ cms.

Also marked on the figure are:

Solar Radius $R_\odot = 6.96 \times 10^{10}$ cm; AU $= 1.5 \times 10^{13}$ cms;

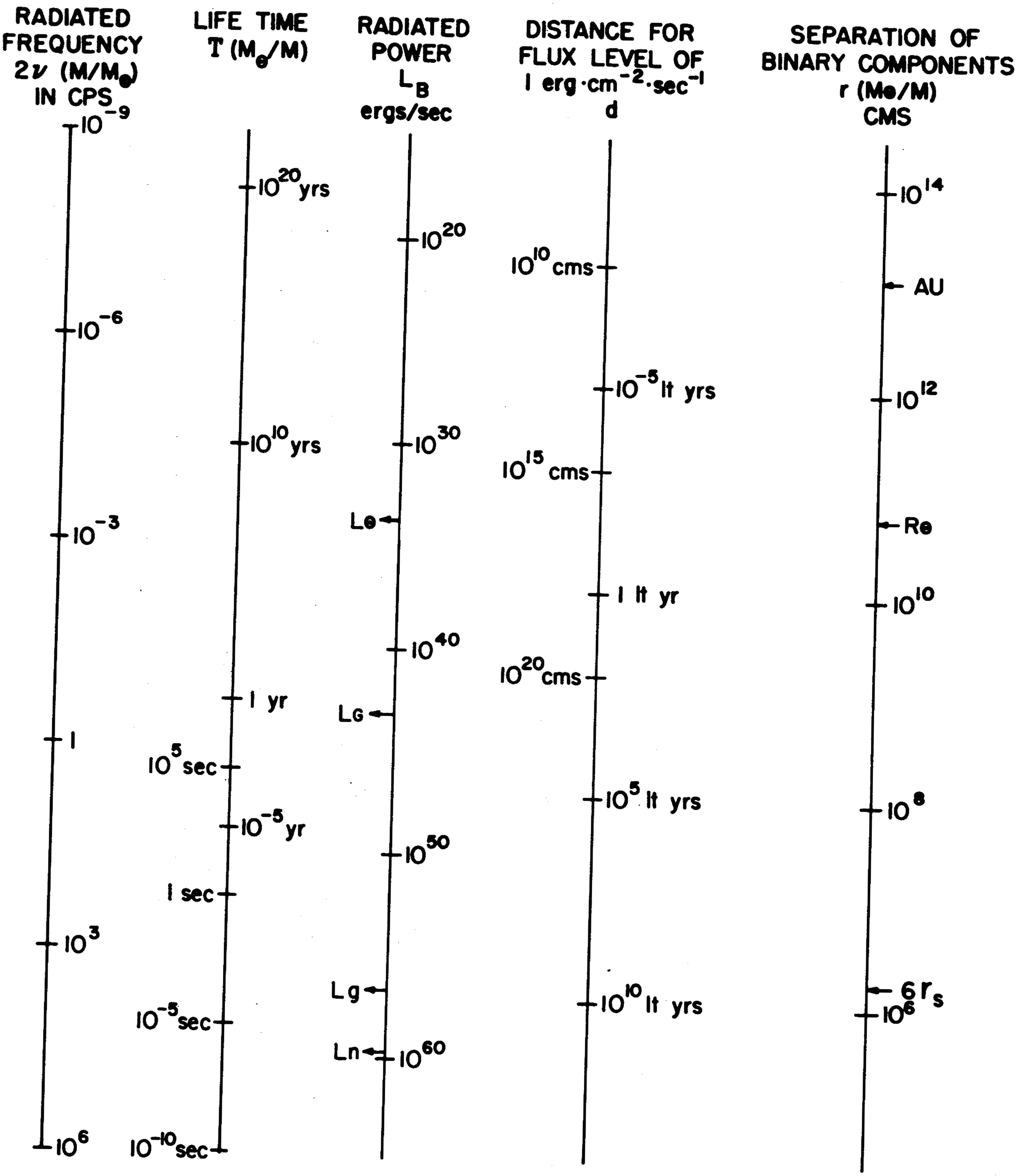
$6r_s = 1.8 \times 10^6$ cms

Solar Luminosity $= L_\odot = 3.9 \times 10^{33}$ ergs $sec^{-1}$

Galactic Luminosity $L_G \cong 10^{43}$ ergs $sec^{-1}$

Gravitational Luminosity $Lg = 3.6 \times 10^{56}$ ergs $sec^{-1}$

Natural unit for Gravitational Radiation $Ln = 3.64 \times 10^{59}$ ergs $sec^{-1}$.



RADIATED FREQUENCY $2\nu$ $(M/M_\odot)$ IN CPS
$10^{-9}$
$10^{-6}$
$10^{-3}$
1
$10^{3}$
$10^{6}$
LIFE TIME $T$ $(M_\odot/M)$
$10^{20}$ yrs
$10^{10}$ yrs
1 yr
$10^{5}$ sec
$10^{-5}$ yr
1 sec
$10^{-5}$ sec
$10^{-10}$ sec
RADIATED POWER $L_B$ ergs/sec
$10^{20}$
$10^{30}$
Le
$10^{40}$
LG
$10^{50}$
Lg
Ln
$10^{60}$
DISTANCE FOR FLUX LEVEL OF 1 $erg \cdot cm^{-2} \cdot sec^{-1}$ d
$10^{10}$ cms
$10^{-5}$ lt yrs
$10^{15}$ cms
1 lt yr
$10^{20}$ cms
$10^{5}$ lt yrs
$10^{10}$ lt yrs
SEPARATION OF BINARY COMPONENTS r $(M_\odot/M)$ CMS
$10^{14}$
AU
$10^{12}$
Re
$10^{10}$
$10^{8}$
$6 r_s$
$10^{6}$

Figure Caption: Gravitational Radiation Data for Two Equal Masses in a Newtonian Circular Orbit. Corresponding quantities are on the same horizontal level in the figure above. In the case where each mass M is one sun's mass $M_\odot$, the figure gives directly the values of the frequency $2\nu$ of the emitted radiation ($\nu$ is the rotation frequency of the binary system), of the distance r between the two masses (orbit diameter) and of the time T remaining before the state r = o is predicted. For other masses, the numbers in the figure give the related quantities as indicated. In every case the figure gives directly the radiated power or gravitational wave luminosity $L_B$, and the distance d at which this radiation could be detected by a receiver capable of detecting 1 erg $cm^{-2}$ $sec^{-1}$ at the appropriate frequency. For $r < 6r_s$ i. e., $r < 12GM/c^2$, these quantities based on linearized theory, are unreliable even as regards order of magnitude. The maximum power emitted by a binary system will likely be somewhat less than the characteristic power Lg, or the maximum distance for 1 erg $cm^{-2}$ $sec^{-1}$ sensitivity detection somewhat less than $5 \cdot 10^9$ light years.